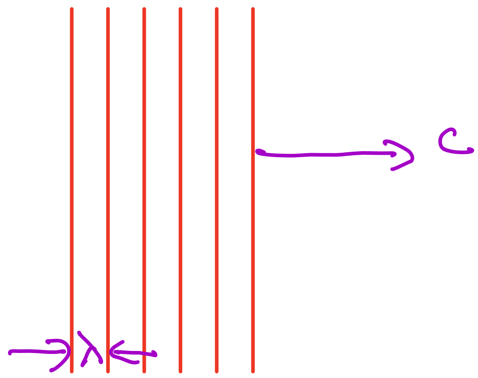
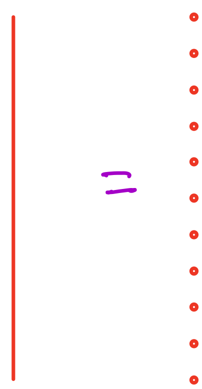
Huygens Principle:

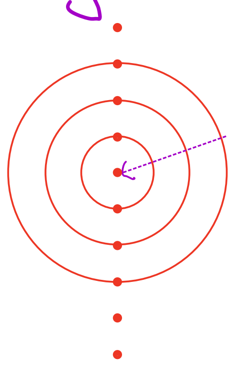
start with light wavefront on a plane wave



can decompose wavefront into "wavelets"

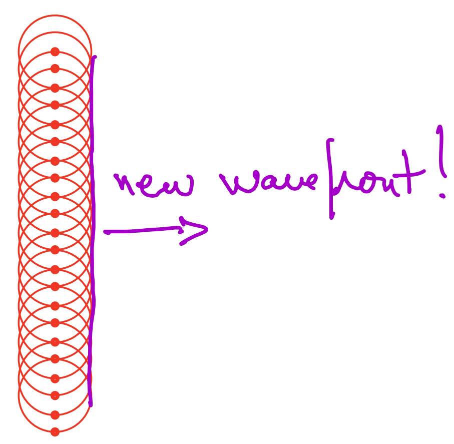


Then let each wavelet send out waves with the same wavelength. in all directions



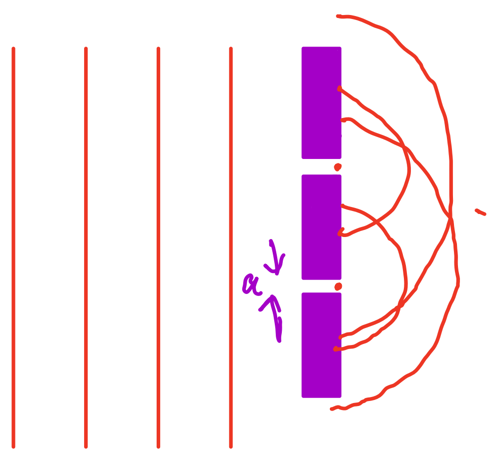
pick one wavelet

now add wavelets from all sources



the ones going backwards are cancelled out by wavelets behind the leading set

This can be used to explain 2-slit interference where width of slit $a << \lambda$



... this reproduces 2-slit interference pattern!

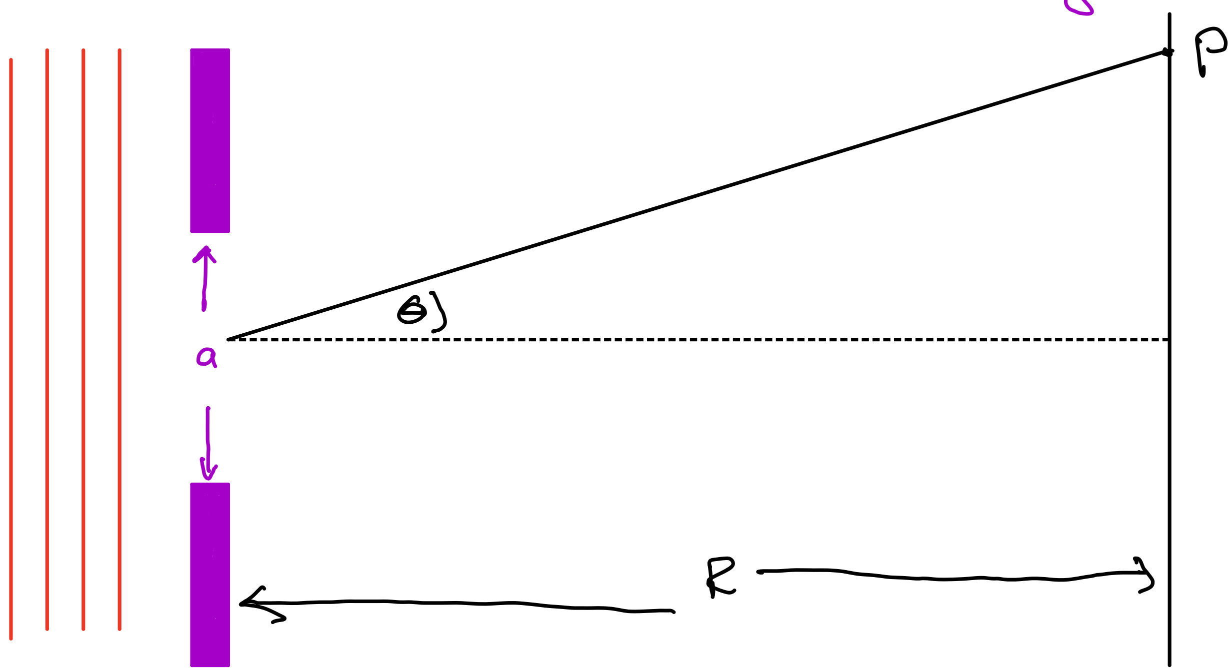
what if slit width $a \sim \lambda$?

Then apply Huygen's principle to all the wavelets that fit inside slit

This is called diffraction $(a > \lambda)$

Fraunhofer diffraction:

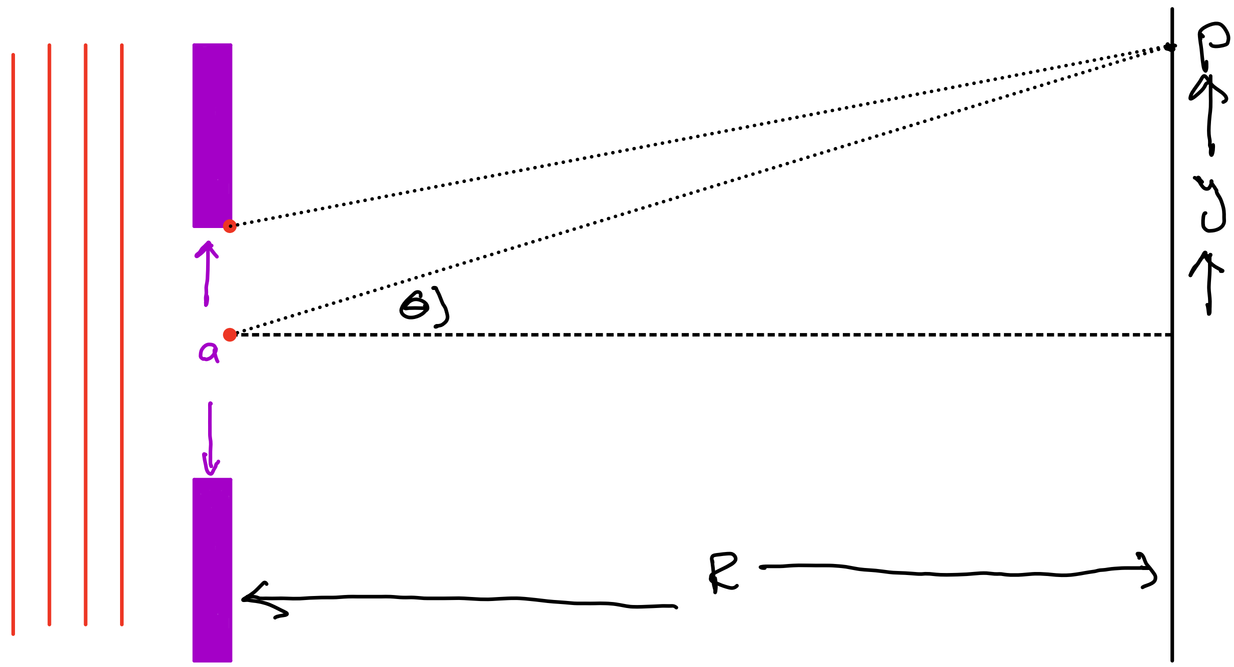
when slit and screen are separated by $R > a$



want to calculate the diffraction (interference) minima and maxima at point P

Diffraction minima

This is where interference is destructive



Start w/2 wavelets: at top of slit and in center

for $R \gg a$, both rays make angle θ wrt dashed line

$\Rightarrow$ this is the same as 2-slit interference, with $d = a/2$: $\Delta r = d\sin\theta = \frac{a}{2}\sin\theta$

for 1st minima, want $\Delta r = \frac{1}{2}\lambda$

so $\frac{a}{2}\sin\theta = \frac{\lambda}{2} \Rightarrow a\sin\theta = \lambda$

$\Rightarrow$ now add another pair of wavelets right below for $R \gg a$, same condition: $a\sin\theta = \lambda$ 1st minima

keep adding pairs: will all contribute same

so final condition for 1st minima:

1st: $a\sin\theta = \pm\lambda$

± because can have a minima <u>below</u> dashed line

call this path difference Δr_1 (center of slit to pt)

2nd minima will occur when $\Delta r_2 = \Delta r_1 + \lambda$

2nd: $a\sin\theta = \pm 2\lambda$

and etc. for n'th minima

$$\boxed{a\sin\theta = \pm n\lambda}$$

Position y on screen given by

$\tan\theta = y/R$

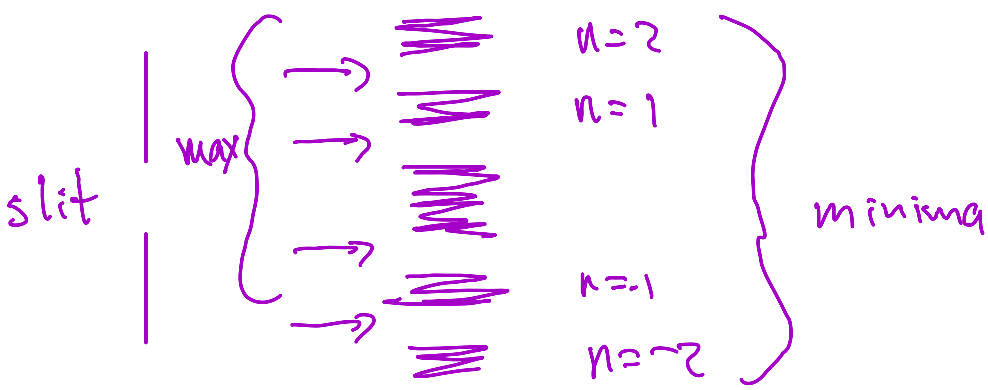
but for $R \gg y$, $\tan\theta = \sin\theta = \frac{\pm n\lambda}{a}$

so $\frac{y}{R} = \frac{n\lambda}{a}$ (y can be + or −)

$$\boxed{y = \frac{n\lambda R}{a}}$$ minima

what about $n=0$? that would mean $\Delta r = 0$ so we would get a bright interference maxima there

interference bands:



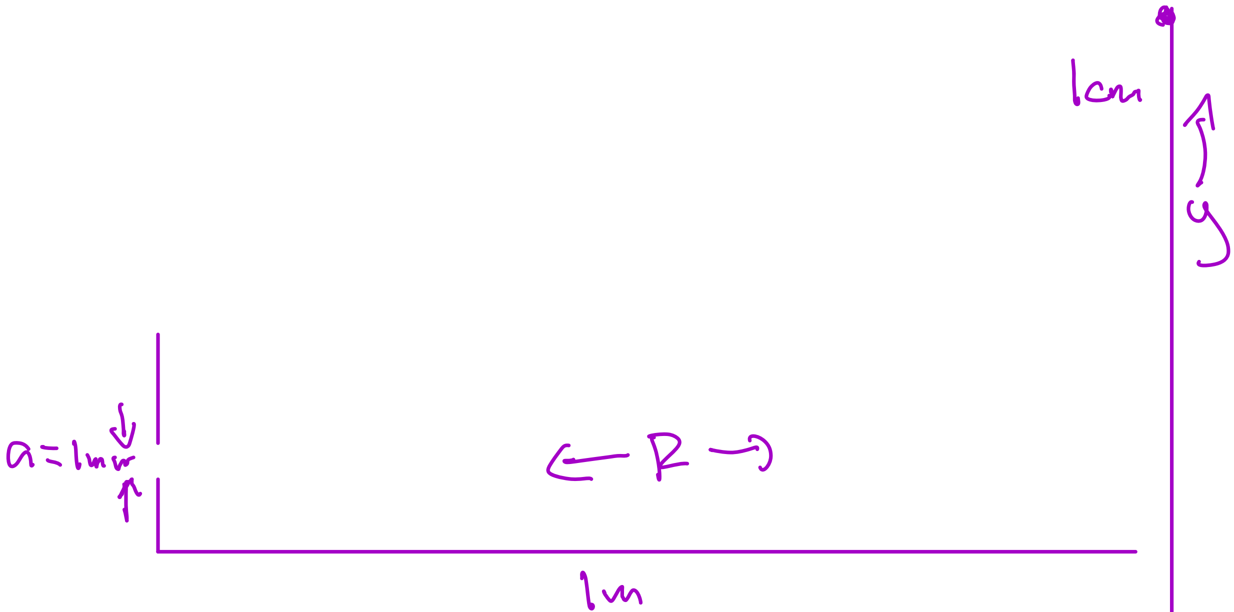
ex: light wavelengths ~300-500 nm

let • $\lambda = 400$ nm

• $a = 1$ mm

• $R = 1$ m

how many diffraction minima are there up to ± 1 cm on screen?



$$y = n\frac{\lambda R}{a} \Rightarrow n = \frac{ya}{\lambda R} = \frac{1\text{cm} \cdot 1\text{mm}}{400\text{nm} \cdot 1\text{m}}$$

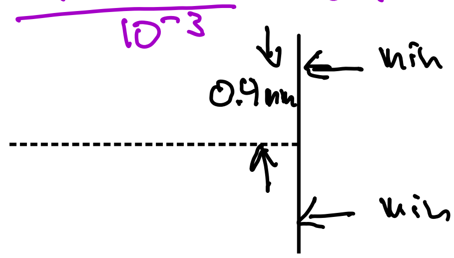
$$= \frac{10^{-2} \cdot 10^{-3}}{0.4 \times 10^{-6} \cdot 1} = \frac{10}{0.4} = 25$$

so 25 minima above, 25 below

what's the width of central max on screen?

1st min is at $y = 1 \cdot \frac{\lambda R}{a} = \frac{400\text{nm} \cdot 1\text{m}}{1\text{mm}}$

$$= \frac{0.4 \times 10^{-6} \cdot 1}{10^{-3}} = 0.4 \times 10^{-3} = 0.4\text{mm}$$



So central max has a width of $2 * 0.4 = 0.8\,mm$

ex: light of wavelength 570nm on slit

screen is $R = 7.5\,m$ away

width of central max is 3.2cm

how wide is slit?

1st minima is $\frac{3.2cm}{2} = 1.6cm$ above center of central max

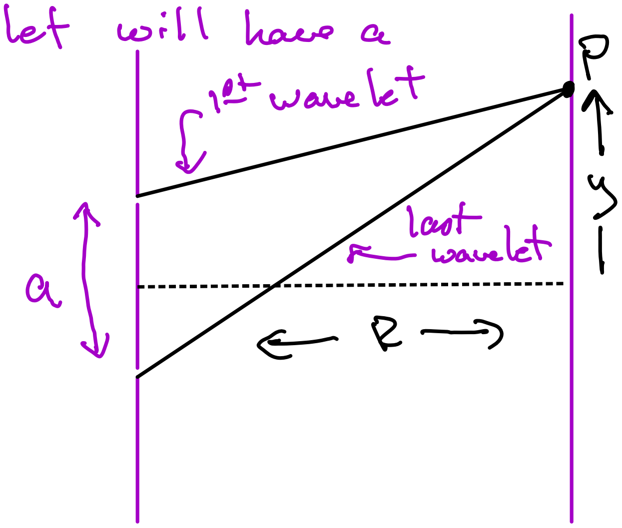
$$\sin\theta \approx \tan\theta = \frac{1.6cm}{7.5m} = 0.00213 = \frac{n\lambda}{a} \quad n = 1$$

$$\text{so } a = \frac{\lambda}{.00213} = \frac{.570 \times 10^{-6}\,m}{2.13 \times 10^{-3}} = 0.27\,mm$$

Intensity pattern

Each wavelet has an E field $\Rightarrow E_n$ n=wavelet

At point P, each wavelet will have a phase shift from having different path length

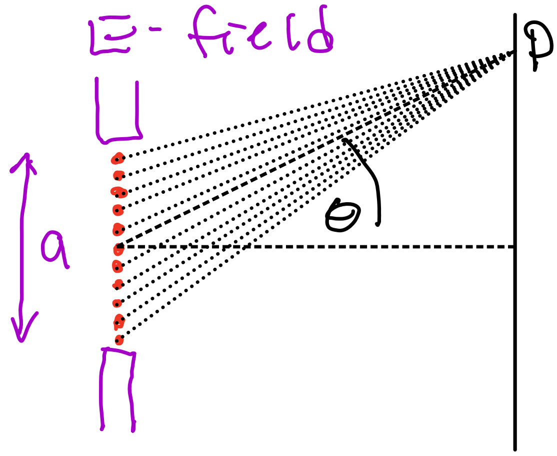


Add up all the waves as vectors

But assum R>>a

$\Rightarrow$ all waves are ~ parallel

so just need to add amplitudes to get final E-field



E_{TOT} at P = sum E at each wavelet

$\Rightarrow$ each E will have slightly different phase at point P due to the different path length

Textbook derivation is pretty good

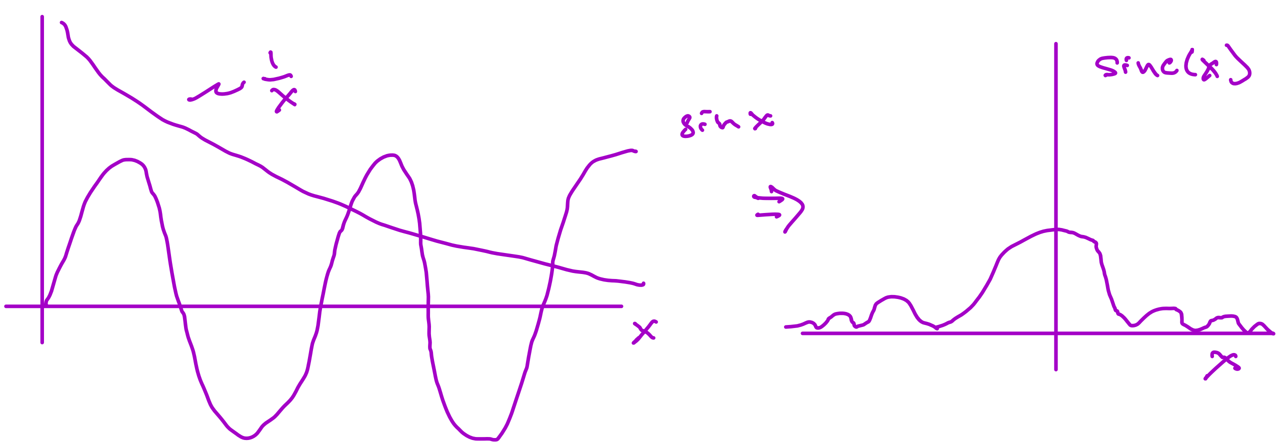
result: $I = I_0 \frac{\sin^2(x)}{x^2}$ where $x = \frac{\pi a \sin\theta}{\lambda}$

where θ = angle between dashed horizontal line (from cent of slit to screen) and wavelet ray from center of slit to P

I_0 is intensity at central maximum: $\theta = 0$

note: $\frac{\sin(x)}{x} \equiv$ "sinc function"

also $\frac{\sin(x)}{x}$ as $x \to 0 = 1$



Intensity minima are when $x = m\pi$

so $\frac{\pi a \sin\theta}{\lambda} = m\pi$

$a \sin\theta = m\lambda$ as before

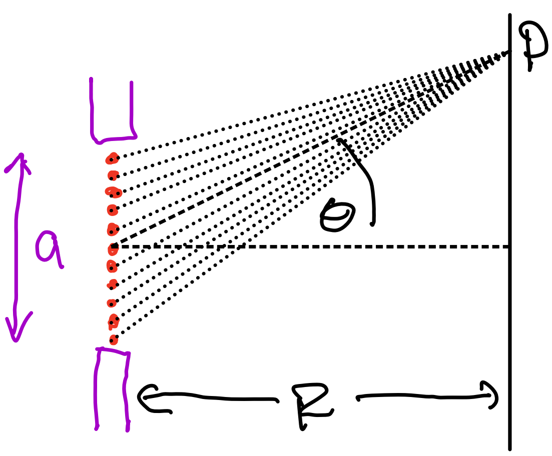
Intensity pattern derivation

To do this right we would pick a point P and add up the interference from all wavelets

First calculate net electric field

⇒ Each wavelet has an E-field that has this form: $E_i = E_0 \cos(kr - \omega t + \phi_i)$

where "i" labels the phase of the wavelet at point P.



$E_T = \sum_i E_i$ sum over wavelets

As # wavelets → ∞, sum turns into an integral:

$$E_T = \frac{1}{a} \int_{-a/2}^{a/2} E_i \, dx$$

the integral goes from $-\frac{a}{2}$ to $\frac{a}{2}$ because we define the phase difference relative to the central wavelet

let x be the distance above the center of the slit, and $-\frac{a}{2} \leq x \leq +\frac{a}{2}$

the phase difference between central wavelet and any other wavelet at coordinate "x"

is as usual: $\phi = k\Delta r$

where Δr = difference in distance from the 2 wavelets to point P

just as w/ 2-slit interference:

$$\Delta r = x \sin\theta$$

so $\phi_i = k\Delta r_i = k x_i \sin\theta$

$$= 2\pi \frac{x_i \sin\theta}{\lambda}$$

integral is:

$$E_{TOT} = \frac{1}{a}\int_{-a/2}^{a/2} E_0 \cos\left(kr - \omega t + \frac{2\pi \sin\theta}{\lambda} x\right) dx$$

- the $\frac{1}{a}$ is needed to cancel out the added position dimension from integrating over x
- or you could think of the integral as being over the fractional distance $d\left(\frac{x}{a}\right)$

this integral is easy:

$$E_{TOT} = \frac{1}{a} E_0 \left. \frac{\sin\left(kr - \omega t + \frac{2\pi}{\lambda}\sin\theta\, x\right)}{\frac{2\pi \sin\theta}{\lambda}} \right|_{-a/2}^{a/2}$$

$$= \frac{E_0 \lambda}{2\pi a \sin\theta} \left[\sin\left(kr - \omega t + \frac{\pi a}{\lambda}\sin\theta\right) - \sin\left(kr - \omega t - \frac{\pi a}{\lambda}\sin\theta\right) \right]$$

let $A = kr - \omega t$, $B = \frac{\pi a}{\lambda}\sin\theta$

then $\sin(A+B) - \sin(A-B)$

$$= \sin A \cos B + \cos A \sin B - (\sin A \cos B - \cos A \sin B)$$

$$= 2 \cos A \sin B$$

so $E_{TOT} = E_0 \cos(kr - \omega t) \dfrac{\sin\left(\frac{\pi a}{\lambda}\sin\theta\right)}{\frac{\pi a \sin\theta}{\lambda}}$

$$\frac{\sin\left(\frac{\pi a}{\lambda}\sin\theta\right)}{\frac{\pi a \sin\theta}{\lambda}} \equiv \operatorname{sinc}(x) \Rightarrow \operatorname{sinc}(x) = \frac{\sin(x)}{x}$$

so $E_{TOT} = E_0 \cos(kr - \omega t)\, \operatorname{sinc}\left(\frac{\pi a \sin\theta}{\lambda}\right)$

then intensity

$$I = \varepsilon_0 E_{TOT}^2 c$$

$$= \underbrace{\epsilon_0 c E_0^2 \cos^2(kr-\omega t)}_{I_0} \operatorname{sinc}^2\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

$$I = I_0 \operatorname{sinc}^2\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

1^{st} min is where $\frac{\pi a \sin\theta}{\lambda} = \pi$ $(\sin(\pi) = 0)$

or $a\sin\theta = \lambda$

2^{nd} min is when $\frac{\pi a \sin\theta}{\lambda} = 2\pi$

or $a\sin\theta = 2\lambda$

etc: $a\sin\theta = \pm m\lambda$ $m = 1, 2, 3 \ldots$ minima

We don't look for maxima this way because the sinc function is the product of 2 functions: $\operatorname{sinc}(x) = \frac{1}{x} \cdot \sin(x)$

max of $\sin(x)$ is not necessarily max of sinc

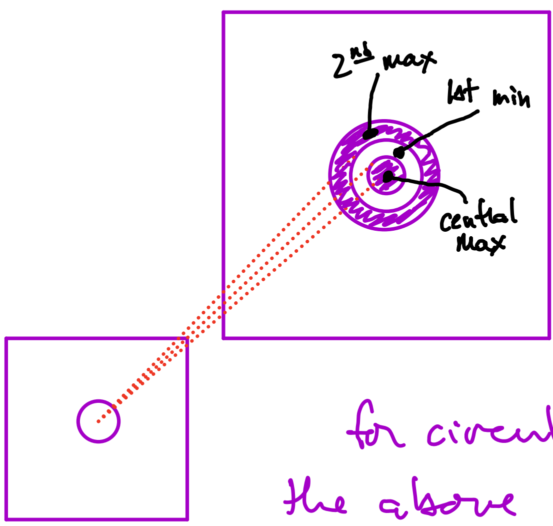
$\Rightarrow$ but min sinc = min sin !

for very small angles $(R \gg a)$ we have minima at

$$a\sin\theta \sim a\theta = n\lambda$$
$$\theta = \frac{n\lambda}{a}$$

this is the angular width of the central max

Circular slits also forms diffraction patterns



central max is called "Airy's disc"

for circular apertures (slits) the above integral is more complicated

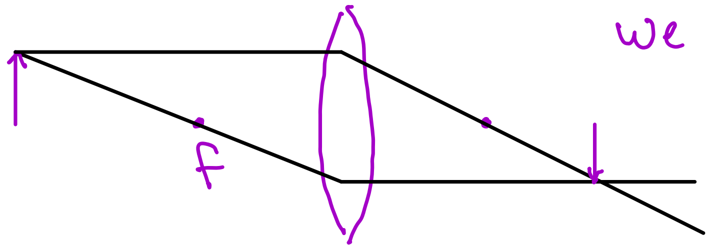
condition for 1st minima:

$$D\sin\theta = 1.22\lambda$$

(1.22 comes from the complicated integral)

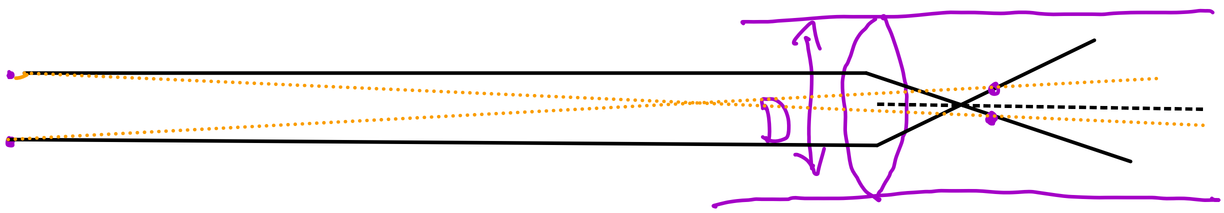
diameter of aperture

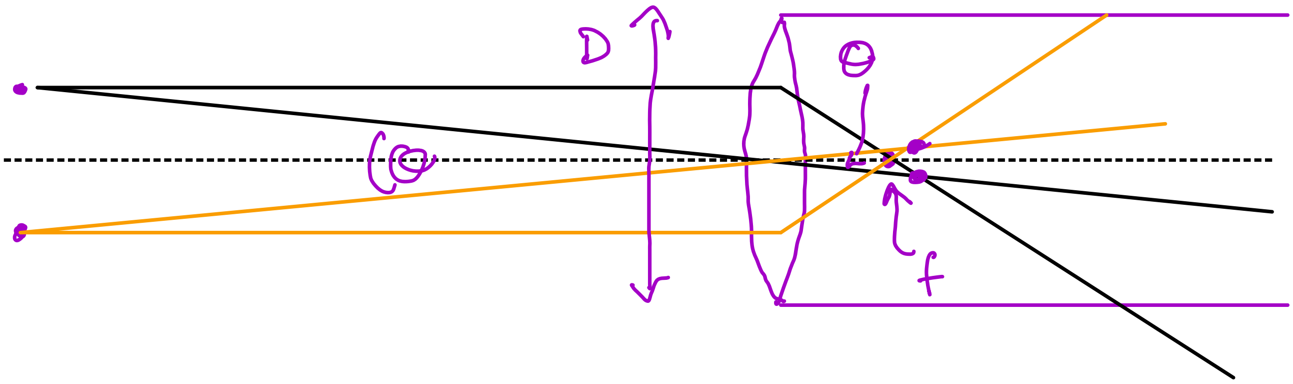
Back to geometric optics & lens



we assume lens focuses to a point but now we know there's diffraction that smears the image out

ex: 2 objects at ∞ (eg. 2 stars) are imaged through a telescope that has aperture D

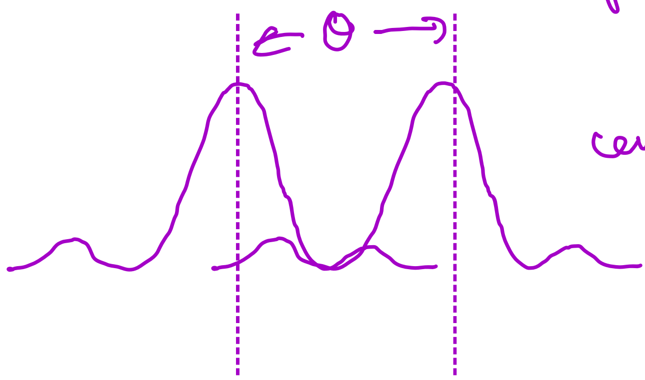




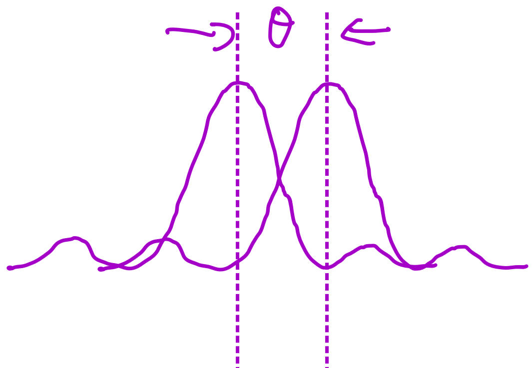
- each object will form an image near focal pt, one below & 1 above
- each image will have a central max due to diffraction through telescope aperture

- let θ be the angular separation between the images → this is also the angular separation of the objects

if $\sin\theta < 1.22\frac{\lambda}{D}$ then the image of one will fall in the Airy disc of the other
⇒ images are not resolvable!



central max are far enough apart so you see the images clearly



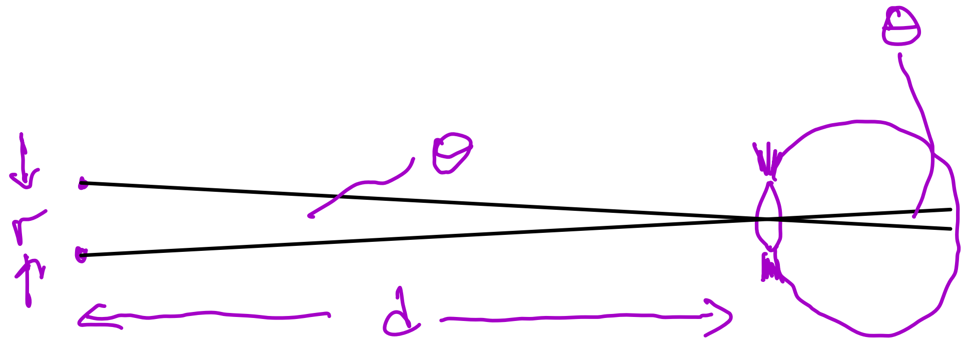
central max's are close - max of 1 falls on min of the other - barely resolvable!

here $\theta = \frac{1.22\lambda}{D}$

if $\theta < \frac{1.22\lambda}{D}$ then the images are not resolvable

⇒ this is the diffraction limit for optical instruments ex: eye pupil can be as small as 2mm

if light has $\lambda = 550\,nm$, what is the minimum angle between 2 objects that you could see?



$$\Theta = \frac{1.22\lambda}{D} = \frac{1.22 * 550\times10^{-9}}{2\times10^{-3}} = 3.36\times10^{-4}$$

if head lights of a car are $r = 1.2m$ apart, what's the farthest dist the car can be for you to still resolve the 2 head lights (and not look like a single headlight)?

$$\Theta = \frac{r}{d} = 3.36\times10^{-4}$$

$$d = \frac{r}{3.36\times10^{-4}} = \frac{1.2m}{3.36\times10^{-4}} = 3577m$$

$\sim 3.6km$

~ 2.2 miles

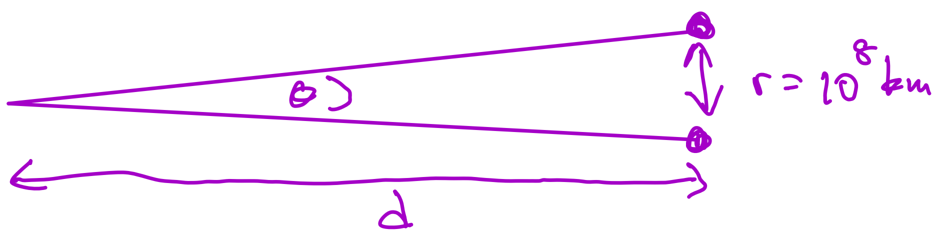
ex: eye pupil is ~ 0.4cm diameter dilated.

if 2 stars are 10^8 km apart (binary stars)

then what's the furthest distance they can be and still viewable by the eye without being "diffraction limited"?

use $\lambda = 400$nm

$$\theta = \frac{1.22\lambda}{D} = \frac{1.22 * 400\times10^{-9}\,m}{0.4\times10^{-2}\,m} = 1.22\times10^{-4}$$



$$\theta = \frac{r}{d} = 1.22\times10^{-4}$$

$$d = \frac{10^8\,km}{1.22\times10^{-4}} = 8.2\times10^{11}\,km = 8.2\times10^{14}\,m$$

note: speed of light $c = 3\times10^8$ m/s

so the time to travel any dist $d = ct$

so the time for light to go 8.2×10^{14} m:

$$t = \frac{d}{c} = \frac{8.2\times10^{14}\,m}{3\times10^8\,m/s} = 2732 \text{ light-sec}$$

$$= 45.5 \text{ light-min}$$

The closest star is Proxima Centauri

$d = 4.246$ light-years

so the eye is incapable of resolving binary stars that are 10^8 km apart

$\Rightarrow$ whats the smallest separation at 4.246 light-yr?

$$\frac{1.22\lambda}{D} = \frac{r}{4.246 \text{ light-yr}}$$

1 light-year = dist light goes in 1 yr = ct

$$= 3\times10^8 \frac{m}{s} * 1\text{yr} * \frac{365\text{d}}{\text{yr}} * \frac{24\text{hr}}{\text{d}} * \frac{3600\text{sec}}{\text{hr}}$$

$$= 9.46\times10^{15} \text{ m}$$

$$r = 1.22 * \frac{400\times10^{-9}}{0.4\times10^{-2}} * 4.246 * 9.46\times10^{15}$$

$$= 4.9\times10^{12} \text{ m}$$

$$\frac{4.9\times10^{12}\text{ m}}{3\times10^8 \frac{m}{s}} = 16335 \text{ light-sec} \cong 272 \text{ light-min}$$

~ 4.5 light-hours

This is past orbit of pluto!

dist earth-sun = 93×10^6 miles $* \frac{5280 ft}{mi} * \frac{1 m}{3.28 ft}$

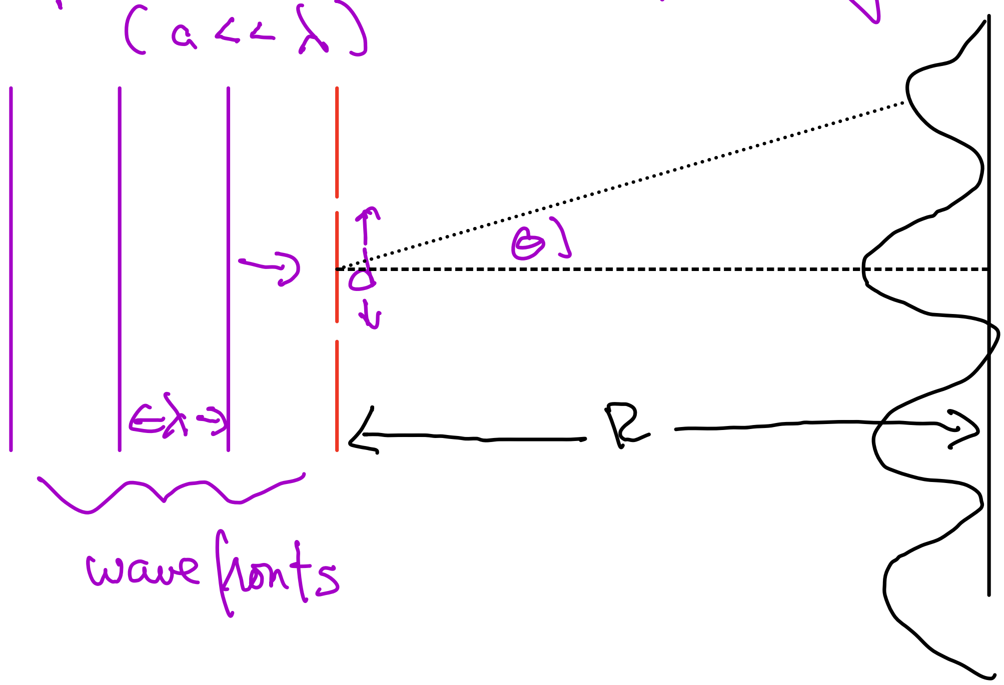
$= 1.5 \times 10^{11}$ m

$\frac{1.5 \times 10^{11} m}{3 \times 10^8 m/s}$ = 500 light-sec ~ 8 light-min

define 1 Astronomical unit (1 AU) = earth-sun dist

Interference + Diffraction

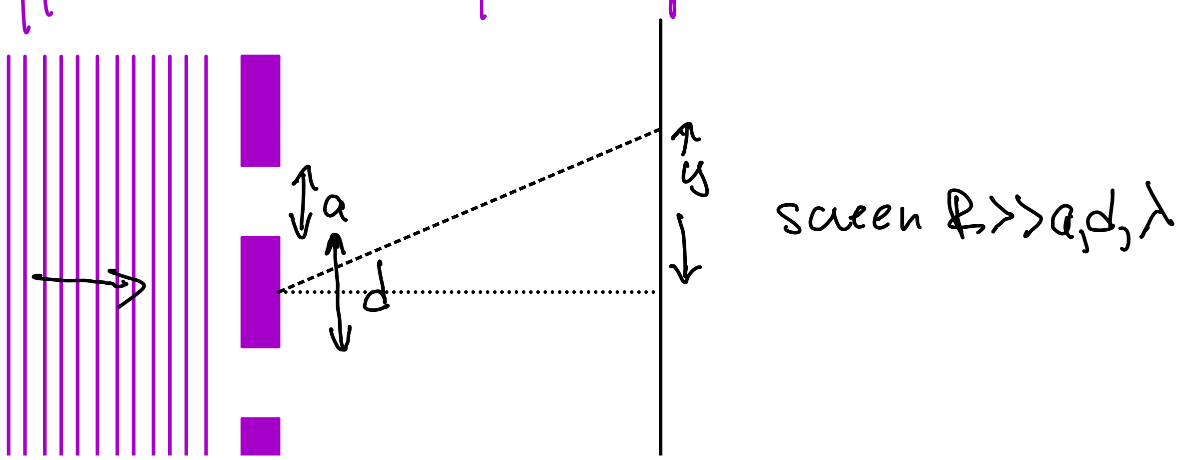
previous 2-slit interference ignored slit width
($a \ll \lambda$)

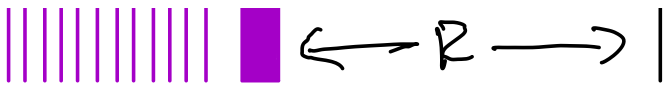


resulting interference pattern had maxima when $d\sin\theta = n\lambda$

$n = 0, \pm 1, \pm 2, \ldots$

If we use light, $\lambda \sim 550\,\text{nm}$, slits are usually with $a \gg \lambda$ so have to include diffraction in interference pattern





2 slit interference: $d\sin\theta = n\lambda$ maxima
$= (n+\frac{1}{2})\lambda$ minima $\Big\}\, n=0,\pm1,\pm2,\ldots$

diffraction: $a\sin\theta = m\lambda$ minima $m=\pm1,\pm2,\ldots$

for $\sin\theta \sim$ small, $\tan\theta \approx \sin\theta \approx \theta$

and $\tan\theta = \dfrac{y}{R}$

so position y for interference min:

$$d\frac{y_n}{R} = (n+\tfrac{1}{2})\lambda$$

or $$y_n = \frac{R\lambda}{d}(n+\tfrac{1}{2})$$

position for diffraction minima:

$$\frac{a\,y_m}{R} = m\lambda$$

or $$y_m = \frac{R\lambda}{a}m$$

distance between interference minima:

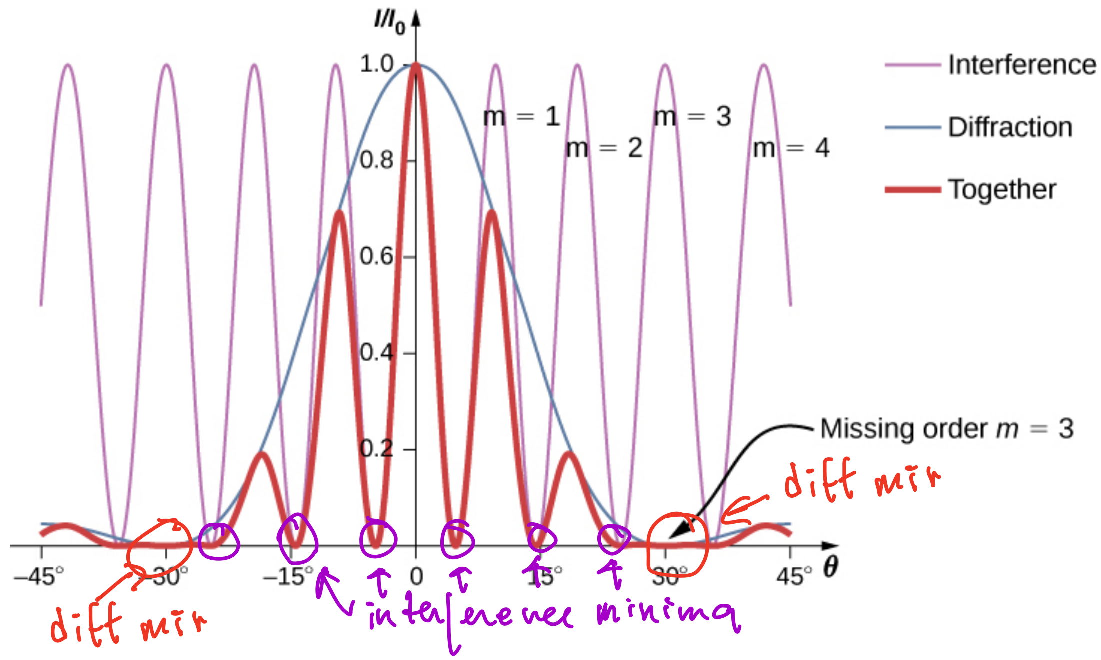
$$\Delta y_i = y_{n+1} - y_n = \frac{R\lambda}{2d}$$

dist between diffraction minima:

$$\Delta y_d = y_{m+1} - y_m = \frac{R\lambda}{a}$$

usually $a < d$ $\therefore$ $\Delta y_i < \Delta y_d$

so there will be many interference minima inside diffraction minima



Intensity pattern for 2 slit:

$I = I_0 \cos^2 \phi/2$ where ϕ = phase diff between waves from top & bot slit

$$\phi = k \Delta r = k * d \sin\theta$$

$$= \frac{2\pi d \sin\theta}{\lambda}$$

Intensity for diffraction of single slit:

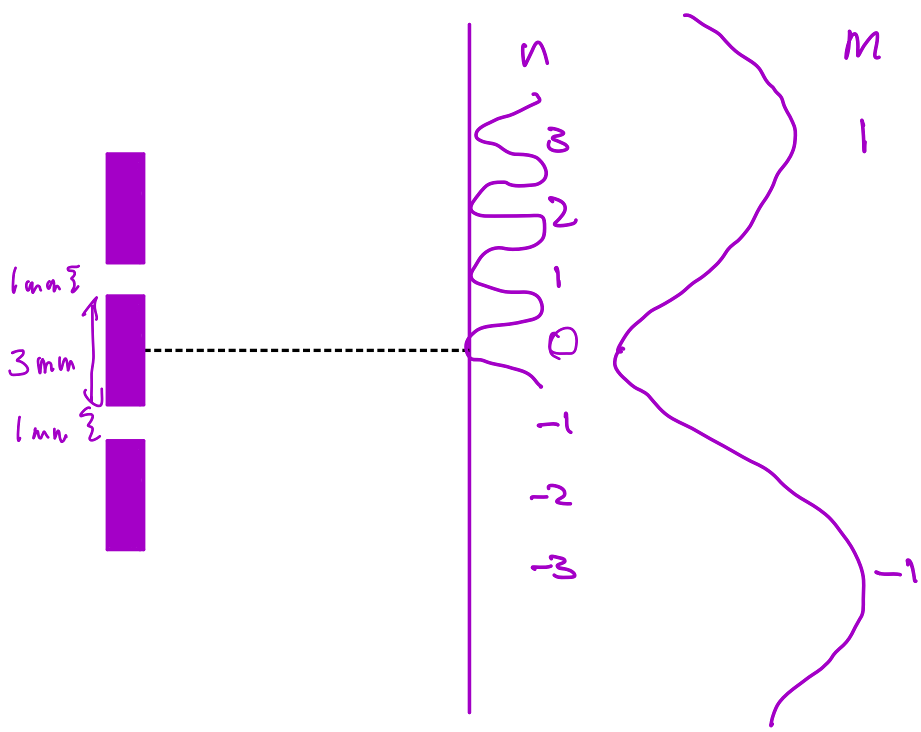
$$I = I_0 \,\mathrm{sinc}^2\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

overall intensity: diffraction modulates interference

$$I = I_0 \underbrace{\cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)}_{\text{interference maxima}} \underbrace{\mathrm{sinc}^2\left(\frac{\pi a \sin\theta}{\lambda}\right)}_{\text{diffraction minima}}$$

when $\frac{d}{a}$ = integer then the diff min will cancel out int max

ex: $a = 1\,mm$, $d = 3\,mm$



interference max is a pts $y_n = n\frac{\lambda R}{d}$

diffraction min " $y_m = m\frac{\lambda R}{a}$

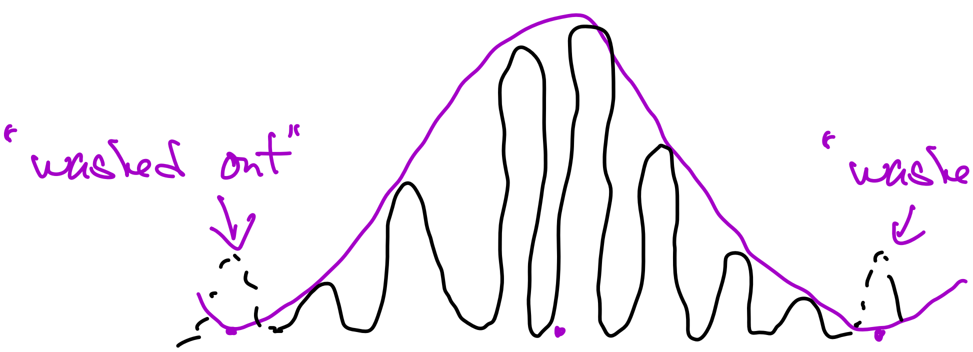
since $d=3a$ can write $y_n = n\frac{\lambda R}{3a}$

So int max coincides w/ diff min when $m = n/3$

this means 3^{rd} interference maxima is washed out by 1^{st} diffraction minima

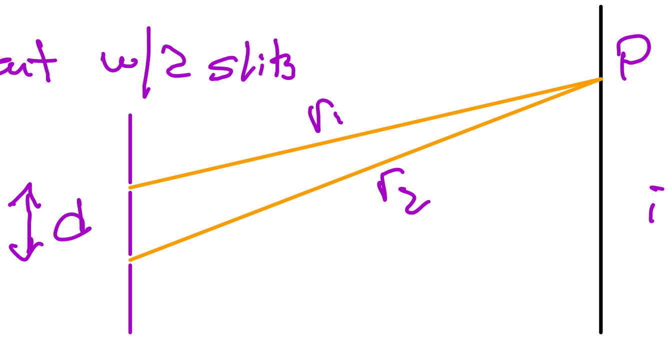
$\Rightarrow$ in general, # interference maxima between diffraction minima will be

$$N = \underbrace{\frac{d}{a}}_{+\text{side}} + \underbrace{\frac{d}{a}}_{-\text{side}} - \underbrace{1}_{\text{central max}} = \frac{2d}{a} - 1 = \frac{2d-a}{a}$$



Multiple slits

start w/2 slits



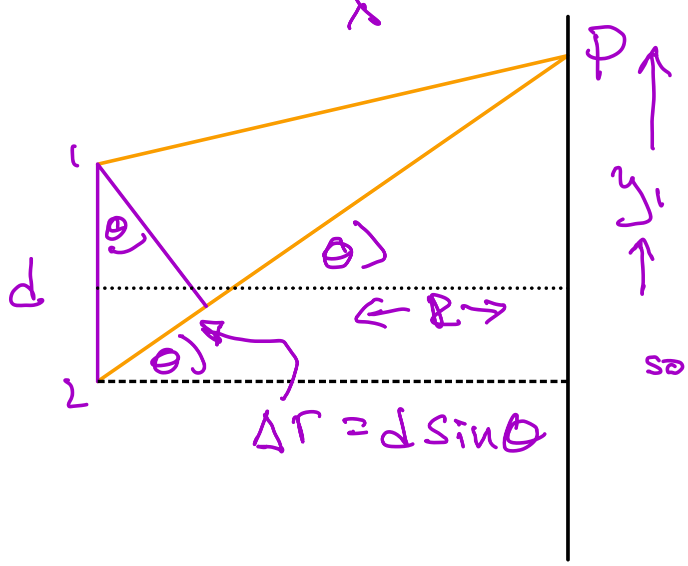
interference is constructive at P if $k\Delta r = 2\pi \cdot n$

$n = 0, \pm 1, \pm 2$, etc.

for $n=1$, then $k\Delta r = 2\pi$ $\quad k = \frac{2\pi}{\lambda}$

and $\Delta r = d\sin\theta$

$$k\Delta r = \frac{2\pi d\sin\theta}{\lambda} = 2\pi \Rightarrow d\sin\theta = \lambda$$



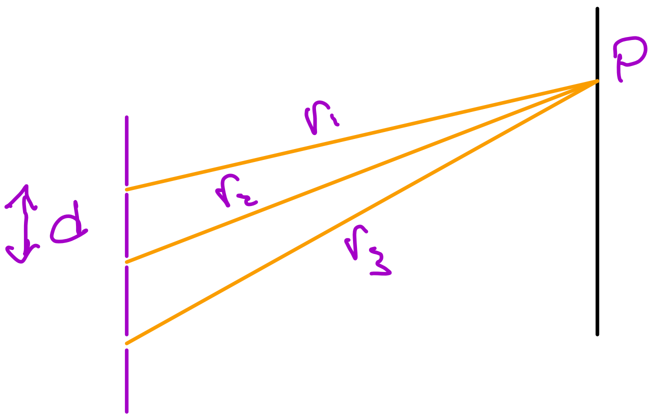
$\tan\theta = \frac{y_1}{R}$

and $\tan\theta \sim \sin\theta$

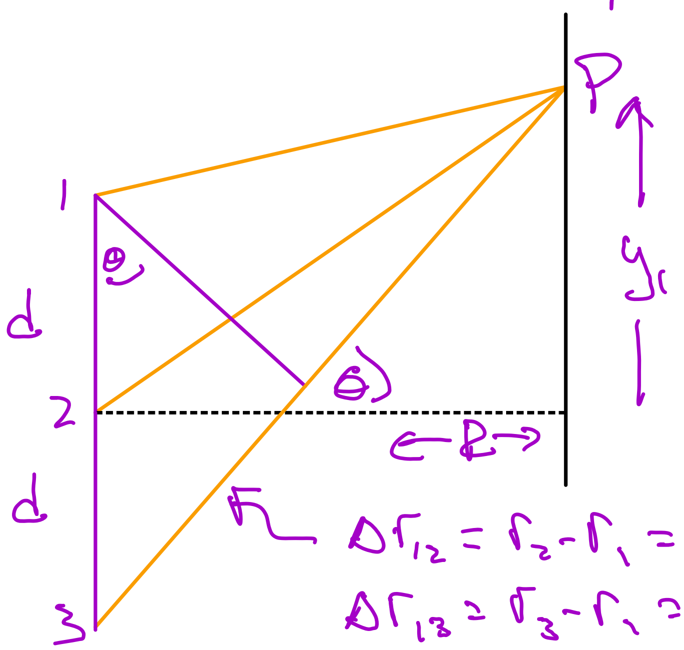
so $\sin\theta = \frac{\lambda}{d} = \frac{y_1}{R}$

$y_1 = \frac{\lambda R}{d}$

y_1 = height above line of symmetry between wave 1 & 2 now add another slit w/same spacing



this wave will also add constructively at P because the path diff to the other 2 waves will also be a multiple of λ



$\Delta r_{12} = r_2 - r_1 = d\sin\theta$

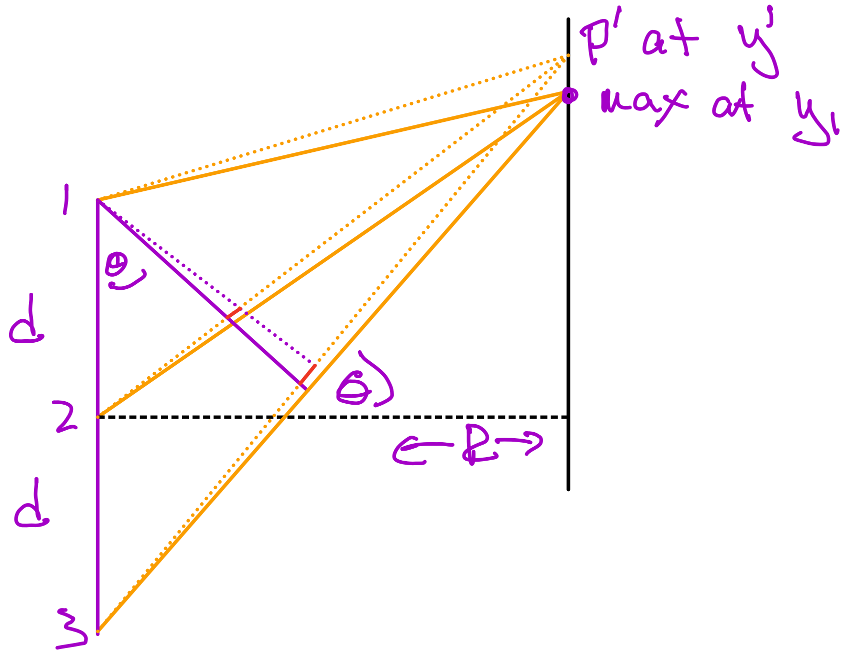
$\Delta r_{13} = r_3 - r_1 = (2d)\sin\theta$

this is in the limit $R >> d$ (R = dist to screen) so that all rays make an angle θ to the horizontal dashed line

Next add more slits. Each slit will add constructively at point P with other sources (at the other slits)

This will produce a very bright max at point P

Now move point P slightly up from the max and add more slits:



Dashed lines are point P', at $y' > y_1$

1st 2 waves have path diff Δr_{12} that is slightly bigger than before (red line)

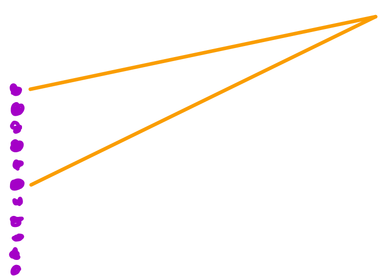
2nd pair has a path difference Δr_{23} that is even bigger than Δr_1

$$\Delta r_{23} > \Delta r_1$$

each additional slit will have an even bigger path difference

for large enough number of slits, at some point the extra path difference will start to be $\frac{1}{2}\lambda$ from the 1st and will cancel each other out

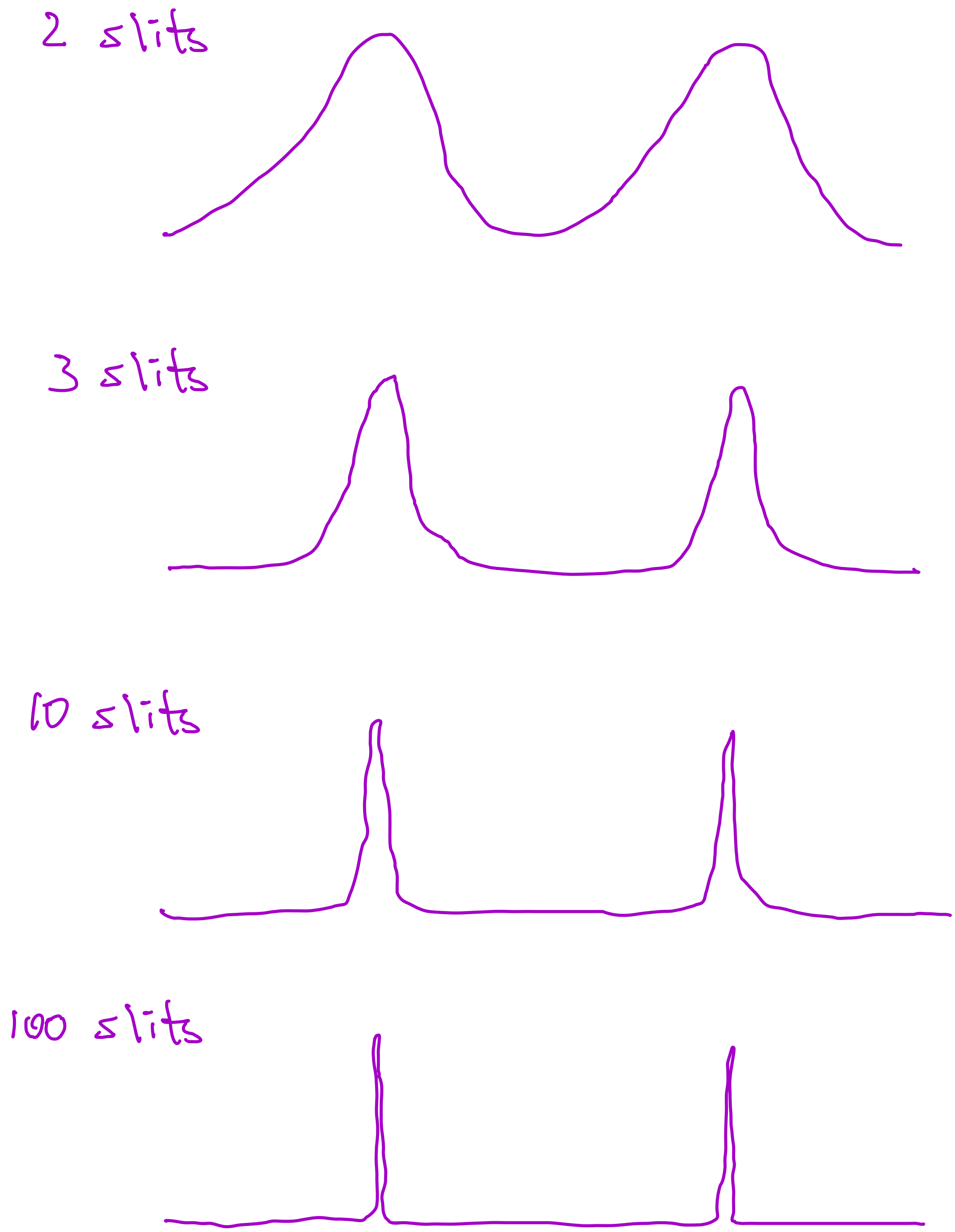
$\Delta r = \frac{1}{2}\lambda$



so right next to 1st max we will fall quickly to zero amplitude due to all the cancellations

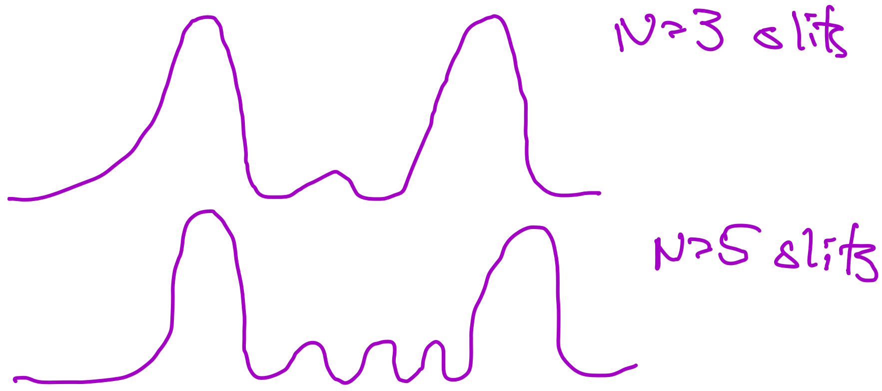
$\Rightarrow$ for m w/ dist d between you will still see max when $d\sin\theta_n = n\lambda$

but as $m \to \infty$ the amplitude falls off more quickly



Actually there is some structure between these maxima but it is much reduced

$\Rightarrow$ there are $N-1$ minima in between maxima for N slits

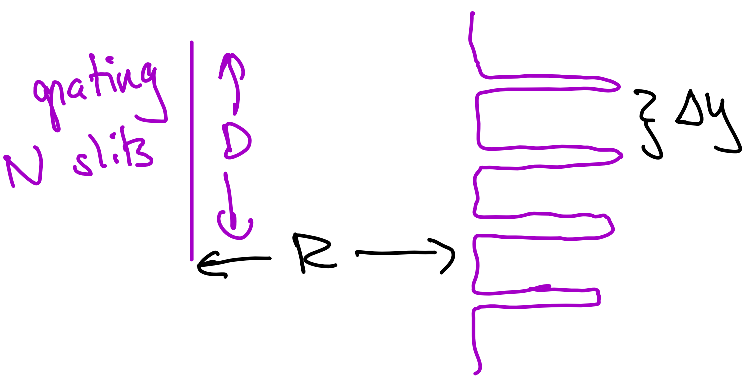


but the slits <u>all</u> have to have the same separation!

<u>Diffraction grating</u>

This is used for very many slits (1000s)

The interference maxima are very peaked!



N slits per length D $\therefore$ spacing $d = \dfrac{D}{N}$

interference max are when $d\sin\theta = n\lambda$

and $\tan\theta \cong \sin\theta = \frac{y_n}{R}$

so $\frac{d\,y_n}{R} = n\lambda$

so $y_n = n\frac{\lambda R}{d}$

then distance between maxima on the screen $\Delta y = y_{n+1} - y_n = \frac{\lambda R}{d}$

so if we measure $\Delta y, d, R$ carefully then $\lambda = \frac{d \cdot \Delta y}{R}$ tells you the wave length of light

$\Rightarrow$ Diffraction gratings can be used to measure wave lengths of light